

N. T. Zinner

# Efimov Trimers near the Zero-crossing of a Feshbach Resonance

December 30, 2011

**Abstract** Near a Feshbach resonance, the two-body scattering length can assume any value. When it approaches zero, the next-order term given by the effective range is known to diverge. We consider the question of whether this divergence (and the vanishing of the scattering length) is accompanied by an anomalous solution of the three-boson Schrödinger equation similar to the one found at infinite scattering length by Efimov. Within a simple zero-range model, we find no such solutions, and conclude that higher-order terms do not support Efimov physics.

## 1 Introduction

In a three-body system of identical bosons, an infinite sequence of bound states appears at the threshold for binding of the two-body subsystems as shown originally by Efimov in the context of nuclear physics [1,2]. However, such states unfortunately do not exist in nuclear system [3]. Efimov states have been a focus of the ultracold gas community since the experimental observations of signatures of such states in a dilute gas of Cesium atoms [4,5,6]. This was followed by studies in three-component  $^6\text{Li}$  [7,8,9,10,11], bosonic  $^7\text{Li}$  [12,13] and  $^{41}\text{K}$  [14], and heteronuclear mixtures of  $^{41}\text{K}$  and  $^{87}\text{Rb}$  [15]. By sweep a Feshbach resonance in  $^7\text{Li}$  over several orders of magnitude [16] a recent study [17] was able to identify eleven distinct features in the recombination rates. Some of these are interpreted as four-body resonances that have been theoretically predicted as associated to the Efimov trimers [18,19,20]. Evidence for potential four-body resonances was also found in [14,21]. The universal theory of four-body states has, however, been questioned [22], and it is unlikely that universality persists to higher particle numbers [23].

The characteristic geometric energy and size scaling of Efimov states near a resonance has been discussed within various models by many people (see [24] and [25] for comprehensive reviews). In cold gases one can use Feshbach resonances to tune interactions [26,27,28,29] and thus reach resonance conditions in a controlled way. This gives the celebrated scaling factor  $e^{-\pi n/s_0} \approx 22.7$  with  $s_0 = 1.00624$ . The energy of the  $n$ th state is then simply related to the first one by  $E(n)/E(0) = 22.7^{-2n}$  and the root-mean-square size by  $r_{rms}(n)/r_{rms}(0) = 22.7^{-n}$ . This result is found within the universal theory where the scattering length is much larger than the range of the interatomic potential. However, evidence for systematic deviations from the universal scaling has been found [14], and therefore it is necessary to consider correction to the universal picture in detail. The experiment reported in Ref. [14] was found to agree qualitatively with the corrections found in [30,31].

In this paper we study the corrections to Efimov physics as the width of the Feshbach resonance becomes smaller. In this case the effective-range is sizable and must be taken into account. The zero-range model is attractive due to its simplicity and range corrections have been discussed in Refs. [31, 32,33,34,35,36]. Here we use a two-channel model to describe the Feshbach resonance [37]. This kind

of model has been considered by several authors for both broad and narrow resonances in few- and many-body contexts [38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53]. A two-channel model gives the energy dependence of the interaction in terms of the background scattering length and the width of the resonance. This is essential since we will be interested in the behavior of the corrections both around the resonance and around the point where the scattering length goes to zero. We recently considered the effects of Feshbach resonances around zero-crossing for a macroscopic Bose-Einstein condensate [54, 55] and found considerable changes in critical particle numbers for very narrow resonances.

Here we consider the interesting question of whether higher order corrections tied to the effective-range term can produce anomalous solutions to the three-boson problem as seen around the resonance. While previous studies using finite-range models [35] and full numerical three-body calculations [36] have considered similar situations, we simplify the discussion considerably by using zero-range model with a two-channel description of the scattering process. This recovers known results about both broad and narrow Feshbach resonances. In addition, we investigate the solutions around the point where the scattering length goes to zero. Here we find no evidence for anomalous solutions that could produce a spectrum similar to the one found by Efimov, regardless of whether the resonance is broad or narrow.

## 2 Feshbach Model

Since we are interested in finite-range corrections within the zero-range approximation, we need to consider multi-channel models. In the case of  $s$ -waves, two-channel models have been derived using various methods, including low-energy effective theory [37], multi-channel quantum-defect theory [56, 57], and resonance models [39]. The on-shell open-open channel  $s$ -wave  $T$ -matrix as a function of magnetic field strength,  $B$ , can be written

$$T(k) = \frac{\frac{4\pi\hbar^2 a_{bg}}{m}}{\left(1 + \frac{\Delta\mu\Delta B}{\frac{\hbar^2 k^2}{m} - \Delta\mu(B - B_0)}\right)^{-1} + ia_{bg}k}, \quad (1)$$

where  $m$  is the atomic mass,  $k$  is the center-of-mass momentum,  $a_{bg}$  is the background  $s$ -wave scattering length,  $B_0$  is the position of the resonance,  $\Delta B$  is the resonance width, and  $\Delta\mu$  is the difference in magnetic moment of the open and closed channels. We can compare this to the usual expression for the  $T$ -matrix in vacuum

$$T(k) = -\frac{\frac{4\pi\hbar^2}{m}}{k \cot \delta(k) - ik} \quad (2)$$

to get an expression for  $k \cot \delta(k)$  as a function of  $B$ . This yields

$$k \cot \delta(k) = -\frac{1}{a_{bg}} \left(1 + \frac{\Delta\mu\Delta B}{\frac{\hbar^2 k^2}{m} - \Delta\mu(B - B_0)}\right)^{-1}. \quad (3)$$

If we neglect the  $k$ -dependent term we recover the usual formula

$$k \cot \delta(k) = -\frac{1}{a(B)} \quad (4)$$

$$a(B) = a_{bg} \left(1 - \frac{\Delta B}{B - B_0}\right). \quad (5)$$

Keeping the  $k$  term and going to the resonance where  $B = B_0$  we find

$$k \cot \delta(k) = -\frac{\frac{1}{2} \frac{2\hbar^2}{m\Delta\mu\Delta B a_{bg}} k^2}{1 + 2 \frac{\hbar^2}{2m\Delta\mu\Delta B a_{bg}} a_{bg} k^2}. \quad (6)$$

If we define  $r_{e0} = -2\hbar^2/m a_{bg} \Delta\mu\Delta B$  and make a low momentum expansion we see that this is consistent with an effective range expansion where the constant term is absent as the  $a(B)$  diverges at  $B = B_0$ .

Notice that  $a_{bg}\Delta\mu\Delta B > 0$  for all Feshbach resonances [29] and therefore  $r_{e0} < 0$  always. Similarly, at zero-crossing where  $B = B_0 + \Delta B$ , we find

$$k \cot \delta(k) = -\frac{1}{a_{bg}} - \frac{2}{a_{bg}^2 r_{e0}} k^{-2}. \quad (7)$$

We thus see that no meaningful effective range expansion is possible. However, the  $T$ -matrix stays finite and one can define an effective potential at zero-crossing that involves  $a_{bg}$  and  $r_{e0}$  [58]. If we introduce the function  $f(B) = (B - B_0)/\Delta B$  we can write the general form

$$k \cot \delta(k) = \frac{1}{a_{bg}} \left[ \frac{f(B) - \frac{1}{2}a_{bg}|r_{e0}|k^2}{1 - f(B) + \frac{1}{2}a_{bg}|r_{e0}|k^2} \right], \quad (8)$$

where  $r_{e0} < 0$  was used. We recover the expression above at resonance where  $f(B_0) = 0$  and zero-crossing where  $f(B_0 + \Delta B) = 1$ . An expansion of the phase-shift to second order gives an expression for the magnetic field dependent effective range of the form [58]

$$r_e(B) = r_{e0} \left[ 1 - \frac{a_{bg}}{a(B)} \right]^2, \quad (9)$$

which clearly shows the divergent nature of  $r_e(B)$  when  $a(B) \rightarrow 0$ . Higher-order corrections do, however, depend on the combination  $a^2 r_e$  which has a finite limit [54,55].

### 3 Effective Hyperspherical Potential

We will employ the hyperspherical formalism [24,25,59], where Efimov states are found from an eigenvalue equation. In the universal limit where only the scattering length is important we have

$$\frac{-\nu \cos(\nu \frac{\pi}{2}) + \frac{8}{\sqrt{3}} \sin(\nu \frac{\pi}{6})}{\sin(\nu \frac{\pi}{2})} = -\frac{\sqrt{2}\rho}{a}. \quad (10)$$

Here  $\rho$  is the hyperspherical radius defined for three identical particles by  $\rho^2 = \frac{1}{3}(r_{12}^2 + r_{13}^2 + r_{23}^2)$  where  $r_{ij}$  is the distance between particles  $i$  and  $j$  [24]. For later convenience, we define the function  $g(\nu)$  to be the left-hand side of Eq. (10), and also

$$g(s) = \frac{-s \cosh(s \frac{\pi}{2}) + \frac{8}{\sqrt{3}} \sinh(s \frac{\pi}{6})}{\sinh(s \frac{\pi}{2})}, \quad (11)$$

which is obtained from the left-hand side of Eq. (10) by the substitution  $\nu = is$ . The solution of this equation for each value of  $\rho$  gives the effective hyperradial potential

$$V_{eff}(\rho) = \frac{\hbar^2}{2m} \frac{\nu(\rho)^2 - \frac{1}{4}}{\rho^2} - Q(\rho), \quad (12)$$

where  $Q(\rho)$  is a diagonal coupling term in the hyperspherical formalism [24]. At the resonance where  $|a| = \infty$ , we find a constant solution  $\nu = is_0$  where  $s_0 = 1.00624$  (which gives the scaling parameter discussed in the introduction). In this case the hyperradial equation for large distances reduces to

$$\left( -\frac{\partial^2}{\partial \rho^2} - \frac{s_0^2 + \frac{1}{4}}{\rho^2} + \kappa^2 \right) f_0(\rho) = 0, \quad (13)$$

where  $-\hbar^2\kappa/2m = E$  is the energy of the three-body state. Here we have neglected  $Q(\rho)$  at large distance; i.e.  $r_0 \ll \rho \ll |a|$  [24], where  $r_0$  is a small distance cut-off (sometimes referred to as a three-body parameter). In the rest of this paper we will be interested in the asymptotic region  $r_0 \ll \rho$ . When there is a two-body bound in the system (as we will assume below), the leading order in  $Q(\rho)$  behaves as  $-1/4\rho^2$  at short distance, while in the absence of such two-body bound state,  $Q(\rho)$  falls off faster

than  $1/\rho^2$  [32,60]. In any case, we are interested in the occurrence of the Efimov effect which hinges on the coefficient of the  $1/\rho^2$  in the region  $r_0 \ll \rho$ , so we neglect the contribution from  $Q(\rho)$  from now on.

Introducing scaled variables  $\tilde{f}_0 = f_0/\sqrt{\rho}$  and  $\tilde{\rho} = \kappa\rho$ , Eq. (13) becomes a Bessel equation. The solution is  $\sqrt{\rho}K_{is_0}(\kappa\rho)$  where  $K$  is the modified Bessel function of the second kind. As the energy goes to zero we recover the Efimov spectrum which arises because  $K_{is_0}(\rho)$  is log-periodic at low energies and thus allows for infinitely many nodes [24]. Below we will be concerned with changes in the Efimov scaling as we vary the parameters of the two-channel Feshbach model.

In the above we have neglected couplings to higher partial waves which are generally small, and we assume that the well-known Thomas collapse is remedied by a boundary condition at small hyperradius;  $f_0(r_0) = 0$ . The latter condition in fact determines the position of the first Efimov state and calibrates the spectrum. It is not clear whether the actual value of  $r_0$  can be determined in the universal theory or not. Recent measurements seem to suggest that  $r_0$  is simply proportional to  $r_{\text{vdW}}$  [61] but the reasons are not entirely understood (see Ref. [62] for a simple argument as to why this should be true).

The next step is to replace the scattering length in Eq. (10) by the full phase shift. In order to do this we have to use the correspondence between  $k$  and the hyperradius  $\rho$  which is  $k \rightarrow k_\rho = \nu(\rho)/(\sqrt{2}\rho)$  [32, 35]. We can then put  $k_\rho \cot \delta(k_\rho)$  on the right-hand side of Eq. (10), where  $\delta(k_\rho)$  is the phase-shift of the scattering of a pure two-particle system evaluated at momentum  $k_\rho$ . This approach was already used in [32,34]. However, as discussed in Ref. [35], it is in fact an approximation. The potential one should use instead is the modified two-body term in the hyperspherical formalism  $V_\rho(r) = V(\sqrt{2}\rho \sin(\frac{r}{\sqrt{2}\rho}))$ . As was shown in [35], for an effective range expansion this gives corrections to the effective potential as compared to using the standard two-body phase shift and the result in fact deviates from the predictions of effective field theory [31]. The corrections to the effective potential are of order  $1/\rho^3$  and in the large distance regime that we are currently interested in we will neglect this and use the simple approximation with  $\delta(k_\rho)$ . We expect that an analysis along the lines of [35] can be made for the two-channel model as well and it would be interesting to see the quantitative nature of this in the future.

Replacing  $-1/a$  in Eq. 10 by Eq. (8) with  $k_\rho = \nu/(\sqrt{2}\rho)$ , we find

$$g(\nu) = \frac{\sqrt{2}\rho}{a_{bg}} \left[ \frac{f(B) - \frac{1}{4}a_{bg}|r_{e0}|\frac{\nu^2}{\rho^2}}{1 - f(B) + \frac{1}{4}a_{bg}|r_{e0}|\frac{\nu^2}{\rho^2}} \right]. \quad (14)$$

We will now proceed to solve this equation for  $\nu(\rho)$  and compare to the universal case for different values of  $B$ . The universal Efimov scaling of energies and radius emerges in the limit of constant  $\nu = is_0$ . When the potential deviates from the constant value it is relevant to ask over what region (if any)  $\nu$  remains essentially flat and what the numerical value is that gives the scaling. The number of bound states is then roughly proportional to  $\ln(\rho_1/\rho_0)$  where  $\rho_0$  and  $\rho_1$  are then boundaries of the flat region [1,2,24].

## 4 Analysis

We will consider the case of  $a_{bg} > 0$  only, the opposite case is similar. This implies that there is a two-body bound state in the scattering channel. This is reflected in the hyperradial potential for  $\rho < a_{bg}$  as discussed in Ref. [32]. Since we are interested in three-body physics only, we will consider the region  $\rho > a_{bg}$  only. Having introduced the function  $f(B)$  above, we can split the analysis of Eq. (14) into two cases, i) near resonance,  $f(B) = 0$ , and ii) near zero-crossing,  $f(B) = 1$ . We proceed to analyse these in turn.

### 4.1 Near resonance

Here we have  $f(B) \rightarrow 0$ . The right-hand side of Eq. (14) becomes

$$\frac{\sqrt{2}\rho}{a_{bg}} \left[ \frac{-\frac{1}{4}\frac{|r_{e0}|}{a_{bg}}\nu^2 \left(\frac{a_{bg}}{\rho}\right)^2}{1 + \frac{1}{4}\frac{|r_{e0}|}{a_{bg}}\nu^2 \left(\frac{a_{bg}}{\rho}\right)^2} \right]. \quad (15)$$

First consider the case  $a_{bg} \gg |r_{e0}|$ . Then for any finite  $\rho$ , Eq. (15) will go to zero. This will also be true for the case where  $\rho \gg a_{bg}$ . Either way we have  $g(\nu) = 0$  which is known to have the anomalous solution  $\nu = is_0$  as discussed above. We thus have Efimov trimers in the region bounded by  $a_{bg} \ll \rho \ll |a|$  as usual. For shorter distances, we have two-body physics at play. In the case  $a_{bg} \sim |r_{e0}|$ , the conclusion is the same.

Next consider the case where  $|r_{e0}| \gg a_{bg}$  which is the case for narrow Feshbach resonances [36, 29]. For any finite  $\rho < |r_{e0}|$ , we find instead of Eq. (15), that the right-hand side of Eq. (14) will behave like  $-\sqrt{2}\rho/a_{bg}$ . Looking at Eq. (11), we see that  $g(s)$  becomes a linear function of  $s$  for  $s \gg 1$ . A solution of the form  $\nu(\rho) = \alpha i\rho/a_{bg}$  can therefore be found in the region  $a_{bg} \ll \rho < |r_{e0}|$ . However, the effective potential in Eq. (12) then gets a constant contribution and this will not produce an Efimov-like spectrum since the requirement is that  $\nu^2$  be constant and negative over an extended region.

The final region to consider is  $\rho \gg |r_{e0}|$ . Here we see from Eq. (15) that we recover  $g(\nu) = 0$  and the condition for the Efimov effect is fulfilled. The region is  $|r_{e0}| \ll \rho < |a|$  as previously found [32, 35, 36]. The zero-range model is thus seen to reproduce this intuitively clear result in a simple and elegant manner.

#### 4.2 Near Zero-crossing

Next we consider the case of zero-crossing where  $a \rightarrow 0$ ,  $f(B) \rightarrow 1$ , and the right-hand side of Eq. (14) becomes

$$4\sqrt{2} \left( \frac{\rho}{a_{bg}} \right)^3 \left[ \frac{1 - \frac{1}{4} \frac{|r_{e0}|}{a_{bg}} \nu^2 \left( \frac{a_{bg}}{\rho} \right)^2}{\frac{|r_{e0}|}{a_{bg}} \nu^2} \right]. \quad (16)$$

First consider the case where  $a_{bg} \gg |r_{e0}|$ . In this case, Eq. (16) blows up because of the  $|r_{e0}|/a_{bg}$  term in the denominator. Neglecting the second term in the numerator (which is small since we are still assuming that  $\rho \gg a_{bg}$ ), we can multiply Eq. (14) by  $\nu^2$  and in turn repeat the argument above about the behavior of  $g(s)$  for large values of  $s$  to obtain a solution that goes like  $\nu \propto i\beta\rho/a_{bg}$  where  $\beta$  contains a factor  $(a_{bg}/|r_{e0}|)^{1/3}$  which we assumed to be large. Again we are lead to the conclusion that the effective hyperradial potential in Eq. (12) will only receive a constant contribution in the region  $|r_{e0}| \ll a_{bg} \ll \rho \ll |f(B) - 1|^{-1}$ . The upper bound is given by how close we are to the point where  $a$  vanishes, similar to the case near resonance where  $a$  is the upper bound. At zero-crossing, the interval becomes unbounded from above, and no Efimov-like potential will emerge anywhere for resonances where  $a_{bg} \gg |r_{e0}|$ . The same is true for  $a_{bg} \sim |r_{e0}|$  by similar analysis.

Now consider instead the opposite limit  $|r_{e0}| \gg a_{bg}$ . For  $\rho \ll |r_{e0}|$ , Eq. (16) becomes simply  $-\sqrt{2}\rho/a_{bg}$ . The solution in this case is again  $\nu \propto i\gamma\rho/a_{bg}$  and nothing interesting happens. The last regime where one can still hope for a non-trivial solution is when  $\rho \gg |r_{e0}|$ , which is where Efimov trimers appear near resonance. However, in similar fashion to the case where  $a_{bg} \gg |r_{e0}|$ , we find that  $\nu \propto i\delta\rho/a_{bg}$  with  $\delta$  containing a factor  $(a_{bg}/|r_{e0}|)^{1/3}$  which is now a small number. This does not, however, change the outcome and no Efimov-like solution is found.

We have done numerical root finding on Eq. (14) to confirm the analysis above. We have tested the numerics and made sure that it reproduces the usual Efimov solution,  $s_0$ , and also that it finds the free solutions [24]. In the case of zero-crossing, we have not found any constant and purely imaginary solution for  $\nu$ . Thus, numerically we also do not see any signs that Efimov-like features are present around zero scattering length.

## 5 Summary

We have considered the occurrence of a geometric spectrum in the three-boson system around Feshbach resonances in ultracold gases when effective-range corrections are included. In particular, we have explored the possibility of having an Efimov-like effect around the zero-crossing where the scattering length goes to zero and higher order effective-range terms must be considered. This is done within a simple zero-range model that allows one to obtain a closed equation for the coefficient of the effective

potential in the hyperspherical approach. The effective potential in turn determines whether there can be universal three-body states with geometric scaling properties.

In the analysis of the closed equation, we recover the conclusions of previous studies, including those obtained through numerical approaches. Close to the resonance, the region of space where universal trimers are supported is bounded from below by either the background scattering length or the background effective range, whichever is larger, while it is bounded from above by the value of the scattering length (which goes to infinite on resonance).

Around the point where the scattering length crosses zero, the zero-range model predicts that no region can be found where the effective potential can support Efimov trimer states. This is true for both broad and narrow Feshbach resonances. Thus even though the magnetic field dependent effective range of a two-channel model diverges around zero-crossing, this does not seem to support a geometric spectrum of Efimov trimers. More generally, this implies that higher-order terms do not add attraction in the three-body channel of a many-body system. This is consistent with the fact that the instabilities observed in condensates with higher-order interaction terms in Refs. [54,55] originate from two-body effects.

## References

1. Efimov, V.: Energy levels arising from resonant two-body forces in a three-body system. *Phys. Lett. B* **33**, 563564 (1970)
2. Efimov, V.: Weakly bound states of three resonantly interacting particles. *Yad. Fiz.* **12**, 1080-1091 (1970); *Sov. J. Nucl. Phys.* **12**, 589-595 (1971)
3. Jensen, A.S., Riisager, K., Fedorov, D.V., Garrido, E.: Structure and reactions of quantum halos. *Rev. Mod. Phys.* **76**, 215-261 (2004)
4. Kraemer, T., Mark, M., Waldburger, P., Danzl, J.G., Chin, C., Engeser, B., Lange, A.D., Pilch, K., Jaakkola, A., Nägerl, H.-C., Grimm, R.: Evidence for Efimov quantum states in an ultracold gas of cesium atoms. *Nature* **440**, 315318 (2006)
5. Knoop, S., Ferlaino, F., Mark, M., Berninger, M., Schöbel, H., Nägerl, H.-C., Grimm, R.: Observation of an Efimov-like trimer resonance in ultracold atom-dimer scattering. *Nat. Phys.* **5**, 227230 (2009)
6. Ferlaino, F., Zenesini, A., Berninger, M., Huang, B., Nägerl, H.-C., Grimm, R.: Efimov Resonances in Ultracold Quantum Gases. *Few-Body Syst* **51**, 113-133 (2011)
7. Ottenstein, T.B., Lompe, T., Kohnen, M., Wenz, A.N., Jochim, S.: Collisional stability of a three-component degenerate Fermi gas. *Phys. Rev. Lett.* **101**, 203202 (2008)
8. Huckans, J.H., Williams, J.R., Hazlett, E.L., Stites, R.W., OHara, K.M.: Three-body recombination in a three-state fermigas with widely tunable interactions. *Phys. Rev. Lett.* **102**, 165302 (2009)
9. Wenz, A.N., Lompe, T., Ottenstein, T.B., Serwane, F., Zrn, G., Jochim, S.: Universal trimer in a three-component Fermi gas. *Phys. Rev. A* **80**, 040702(R) (2009)
10. Nakajima, S., Horikoshi, M., Mukaiyama, T., Naidon, P., Ueda, M.: Nonuniversal Efimov atom-dimer resonances in a three-component mixture of  $^6\text{Li}$ . *Phys. Rev. Lett.* **105**, 023201 (2010)
11. Nakajima, S., Horikoshi, M., Mukaiyama, T., Naidon, P., Ueda, M.: Measurement of an Efimov trimer binding energy in a three-component mixture of  $^6\text{Li}$ . *Phys. Rev. Lett.* **106**, 143201 (2011)
12. Gross, N., Shotan, Z., Kokkelmans, S., Khaykovich, L.: Observation of universality in ultracold  $^7\text{Li}$  three-body recombination. *Phys. Rev. Lett.* **103**, 163202 (2009)
13. Gross, N., Shotan, Z., Kokkelmans, S., Khaykovich, L.: Nuclear-spin-independent short-range three-body physics in ultracold atoms. *Phys. Rev. Lett.* **105**, 103203 (2010)
14. Zaccanti, M., Deissler, B., D'Errico, C., Fattori, M., Jona-Lasinio, M., Mller, S., Roati, G., Inguscio, M., Modugno, G.: Observation of an Efimov spectrum in an atomic system. *Nat. Phys.* **5**, 586 (2009)
15. Barontini, G., Weber, C., Rabatti, F., Catani, J., Thalhammer, G., Inguscio, M., Minardi, F.: Observation of heteronuclear atomic Efimov resonances. *Phys. Rev. Lett.* **103**, 043201 (2009)
16. Pollack, S.E., Dries, D., Junker, M., Chen, Y.P., Corcovilos, T.A., Hulet, R.G.: Extreme Tunability of Interactions in a  $^7\text{Li}$  Bose-Einstein Condensate. *Phys. Rev. Lett.* **102**, 090402 (2009).
17. Pollack, S.E., Dries, D., Hulet, R.G.: Universality in three- and four-body bound states of ultracold atoms. *Science* **326**, 1683-1686 (2009)
18. Platter, L., Hammer, H.-W., Meißner, U.-G.: Four-boson system with short-range interactions. *Phys. Rev. A* **70**, 052101 (2004)
19. Hanna, G.J., Blume, D.: Energetics and structural properties of three-dimensional bosonic clusters near threshold. *Phys. Rev. A* **74**, 063604 (2006)
20. von Stecher, J., DIncao, J.P., Greene, C.H.: Signatures of universal four-body phenomena and their relation to the Efimov effect. *Nat. Phys.* **5**, 417421 (2009)
21. Ferlaino, F., Knoop, S., Berninger, M., Harm, W., DIncao, J.P., Ngerl, H.-C., Grimm, R.: Evidence for universal four-body states tied to an Efimov trimer. *Phys. Rev. Lett.* **102**, 140401 (2009)
22. Yamashita, M.T., Tomio, L., Delfino, A., Frederico, T.: Four-boson scale near a Feshbach resonance. *Europhys. Lett.* **75**, 555-561 (2006)



23. Yamashita, M.T., Fedorov, D.V., Jensen, A.S.: Universality of Brunnian ( $N$ -body Borromean) four- and five-body systems. *Phys. Rev. A* **81**, 063607 (2010)
24. Nielsen, E., Fedorov, D.V., Jensen, A.S., Garrido, E.: The three-body problem with short-range interactions. *Phys. Rep.* **347**, 373-459 (2001)
25. Braaten, E., Hammer, H.-W.: Universality in few-body systems with large scattering length. *Phys. Rep.* **428**, 259390 (2006)
26. Timmermans, E., Tommasini, P., Hussein, M., Kerman, A.: Feshbach resonances in atomic BoseEinstein condensates. *Phys. Rep.* **315**, 199-230 (1999)
27. Bloch, I., Dalibard, J., Zwerger, W.: Many-body physics with ultracold gases. *Rev. Mod. Phys.* **80**, 885-964 (2008)
28. Giorgini, S., Pitaevskii, L.P., Stringari, S.: Theory of ultracold atomic Fermi gases. *Rev. Mod. Phys.* **80**, 1215-1274 (2008)
29. Chin, C., Grimm, R., Julienne, P.S., Tiesinga, E.: Feshbach resonances in ultracold gases. *Rev. Mod. Phys.* **82**, 1225-1286 (2010)
30. Thøgersen, M., Fedorov, D.V., Jensen, A.S.: Universal properties of Efimov physics beyond the scattering length. *Phys. Rev. A* **78**, 020501(R) (2008)
31. Platter, L., Ji, C., Phillips, D.R.: Range corrections to three-body observables near a Feshbach resonance. *Phys. Rev. A* **79**, 022702 (2009)
32. Fedorov, D.V., Jensen, A.S.: Regularization of a three-body problem with zero-range potentials. *J. Phys. A: Math. Gen.* **34**, 6003-6012 (2001)
33. Fedorov, D.V., Jensen, A.S.: Contact Interaction Model for Three-Body Systems. *Few-Body Syst* **31**, 229-234 (2002)
34. Jonsell, S.: Universality of the three-boson system close to a Feshbach resonance. *J. Phys. B: At. Mol. Opt. Phys.* **37**, S245-S255 (2004)
35. Thøgersen, M., Fedorov, D.V., Jensen, A.S., Esry, B.D., Wang, Y.: Conditions for Efimov physics for finite-range potentials. *Phys. Rev. A* **80**, 013608 (2009)
36. Wang, Y., DIncao, J.P., Esry, B.D.: Ultracold three-body collisions near narrow Feshbach resonances. *Phys. Rev. A* **83**, 042710 (2011)
37. Bruun, G.M., Jackson, A.D., Kolomeitsev, E.E.: Multichannel scattering and Feshbach resonances: Effective theory, phenomenology, and many-body effects. *Phys. Rev. A* **71**, 052713 (2005)
38. Phillips, D.R., Beane, S.R., Cohen, T.D.: Non-perturbative regularization and renormalization: simple examples from non-relativistic quantum mechanics. *Annals of Physics* **263**, 255-275 (1998).
39. Kokkelmans, S.J.J.M.F., Milstein, J.N., Chiofalo, M.L., Walser, R., Holland, M.J.: Resonance superfluidity: Renormalization of resonance scattering theory. *Phys. Rev. A* **65**, 053617 (2002)
40. Petrov, D.S.: Three-boson problem near a narrow Feshbach resonance. *Phys. Rev. Lett.* **93**, 143201 (2004)
41. Jonsell, S.: Efimov states for systems with negative scattering lengths. *Europhys. Lett.* **46**, 8-14 (2006)
42. Massignan, P., Castin, Y.: Three-dimensional strong localization of matter waves by scattering from atoms in a lattice with a confinement-induced resonance. *Phys. Rev. A* **74**, 013616 (2006)
43. Diener, R.B., Ho, T.L.: Fermions in Optical Lattices Swept across Feshbach Resonances. *Phys. Rev. Lett.* **96**, 010402 (2006)
44. Collin, A., Massignan, P., Pethick, C.J.: Energy-dependent effective interactions for dilute many-body systems. *Phys. Rev. A* **75**, 013615 (2007)
45. Braaten, E., Kusunoki, M., Zhang, D.: Scattering Models for Ultracold Atoms. *Annals Phys.* **323**, 1770-1815 (2008)
46. Gogolin, A.O., Mora, C., Egger, R.: Analytical Solution of the Bosonic Three-Body Problem. *Phys. Rev. Lett.* **100**, 140404 (2010)
47. Massignan, P., Stoof, H.T.C.: Efimov states near a Feshbach resonance. *Phys. Rev. A* **78**, 030701(R) (2008)
48. Platter, L., Ji, C., Phillips, D.: Range corrections to three-body observables near a Feshbach resonance. *Phys. Rev. A* **79**, 022702 (2009)
49. Pricoupenko, L.: Crossover in the Efimov spectrum. *Phys. Rev. A* **82**, 043633 (2010)
50. Jona-Lasinio, M., Pricoupenko, L.: Three Resonant Ultracold Bosons: Off-Resonance Effects. *Phys. Rev. Lett.* **104**, 023201 (2010)
51. Pricoupenko, L., Jona-Lasinio, M.: Ultracold bosons in the vicinity of a narrow resonance: Shallow dimer and recombination. *Phys. Rev. A* **84**, 062712 (2011)
52. Castin, Y., Tignone, E.: Trimers in the resonant (2+1)-fermion problem on a narrow Feshbach resonance: Crossover from Efimovian to hydrogenoid spectrum. *Phys. Rev. A* **84**, 062704 (2011)
53. Naidon, P., Ueda, M.: The Efimov effect in lithium 6. *Comptes Rendus Physique* **12**, 13-26 (2011)
54. Zinner, N.T., Thøgersen, M.: Stability of a Bose-Einstein condensate with higher-order interactions near a Feshbach resonance. *Phys. Rev. A* **80**, 023607 (2009)
55. Thøgersen, M., Zinner, N.T., Jensen, A.S.: Thomas-Fermi approximation for a condensate with higher-order interactions. *Phys. Rev. A* **80**, 043625 (2009)
56. Mies, F.H., Julienne, P.S.: A multichannel quantum defect analysis of two-state couplings in diatomic molecules. *J. Chem Phys.* **80**, 2526-2536 (1984)
57. Julienne, P.S., Gao, B.: Simple Theoretical Models for Resonant Cold Atom Interactions. *AIP Conf. Proc.* **869**, 261-268 (2006)
58. Zinner, N.T.: Effective Potential for Ultracold Atoms at the Zero-Crossing of a Feshbach Resonance. *arXiv:0909.1314v4* (2009)
59. Fedorov, D.V., Jensen, A.S.: Efimov effect in coordinate space Faddeev equations. *Phys. Rev. Lett.* **71**, 4103 (1993)

- 60. Nielsen, E., Macek, J.H.: Low-energy recombination of identical bosons by three-body collisions. *Phys. Rev. Lett.* **83**, 1566–1569 (1999)
- 61. Berninger, M., Zenesini, A., Huang, B., Harm, W., Nägerl, H.-C., Ferlaino, F., Grimm, R., Julienne, P.S., Hutson, J.M.: Universality of the three-body parameter for Efimov states in ultracold cesium. *Phys. Rev. Lett.* **107**, 120401 (2011)
- 62. Chin, C.: Universal scaling of Efimov resonance positions in cold atom systems. *arXiv:1111.1484v2* (2011)